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Solution of a bisymmetry equation on a restricted domain

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This paper is dedicated to the 70th birthday of Professor Zoltán Daróczy

Abstract. Let $X \subset \mathbb{R}$ be an open interval and define the set Δ by $\Delta = \{(x, y) \in X \times X \mid x \leq y\}$. In this note we give the solution of the equation $F(G(x, y), G(u, v)) = G(F(x, u), F(y, v))$, which holds for all $(x, y) \in \Delta$, $(x, u) \in \Delta$, $(y, v) \in \Delta$, and $(u, v) \in \Delta$, where the functions $F : \Delta \to X$ and $G : \Delta \to X$ are continuous and strictly increasing in each variable, and we suppose that $F(x, x) = x$ and $G(x, x) = x$ for all $x \in X$. The problem has been posed and investigated by M. V. Sokolov in [6].

1. Introduction

In the following we denote the set of real numbers and the set positive integers by $\mathbb{R}$ and $\mathbb{N}$, respectively. By an interval we mean a subinterval of positive length of $\mathbb{R}$ (possibly unbounded) and by a rectangle we mean the Cartesian product of two intervals. A real-valued continuous function defined on an interval or on a rectangle is called CM function if it is strictly monotonic in each variable and called CI function if it is strictly increasing in each variable.

Let I and J be intervals, and let R be a rectangle such that $I \times J \subset R$. A function $Q : R \to \mathbb{R}$ is a quasi-sum on $I \times J$ if there exist CM functions $\alpha : I \to \mathbb{R}$, $\beta : J \to \mathbb{R}$ and $\gamma : \alpha(I) + \beta(J) \to \mathbb{R}$ such that $Q(x, y) = \gamma(\alpha(x) + \beta(y))$ for all $(x, y) \in I \times J$. The triple (α, β, γ) is called a generator of Q (functions α, β, and

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γ are generator functions of Q). A function $Q : R \to \mathbb{R}$ is a local quasi-sum on $I \times J$ if for all $(i, j) \in I \times J$ there exist an open rectangle R_0 such that $(i, j) \in R_0$ and Q is a quasi-sum on $(I \times J) \cap R_0$.

The equation of generalized bisymmetry

$$F(G_1(x, y), G_2(u, v)) = G(F_1(x, u), F_2(y, v)), \tag{B}$$

where the functions are defined on rectangles was investigated by several authors (see e.g. Aczél [1] and Maksa [2]). The CM solutions of equation (B) are given by the following

Theorem 1.1 (Maksa [2]). *Let X_{11}, X_{12}, X_{21}, and X_{22} be intervals and let $F_1 : X_{11}\times X_{12} \to \mathbb{R}$, $F_2 : X_{21}\times X_{22} \to \mathbb{R}$, $G_1 : X_{11}\times X_{21} \to \mathbb{R}$, $G_2 : X_{12}\times X_{22} \to \mathbb{R}$, $F : G_1(X_{11}, X_{21}) \times G_2(X_{12}, X_{22}) \to \mathbb{R}$, $G : F_1(X_{11}, X_{12}) \times F_2(X_{21}, X_{22}) \to \mathbb{R}$ be CM functions. Equation* (B) *holds for all $(x, y, u, v) \in X_{11} \times X_{21} \times X_{12} \times X_{22}$ if, and only if, there exists an interval I and there exist CM functions $\varphi : I \to \mathbb{R}$, $\alpha_1 : G_1(X_{11}, X_{21}) \to \mathbb{R}, \alpha_2 : G_2(X_{12}, X_{22}) \to \mathbb{R}, \gamma_1 : F_1(X_{11}, X_{12}) \to \mathbb{R}, \gamma_2 : F_2(X_{21}, X_{22}) \to \mathbb{R}, \beta_{11} : X_{11} \to \mathbb{R}, \beta_{12} : X_{12} \to \mathbb{R}, \beta_{21} : X_{21} \to \mathbb{R}$, and $\beta_{22} : X_{22} \to \mathbb{R}$ such that*

$$\begin{aligned}
F(x, y) &= \varphi^{-1}(\alpha_1(x) + \alpha_2(y)), && (x, y) \in G_1(X_{11}, X_{21}) \times G_2(X_{12}, X_{22})\\
F_1(x, y) &= \gamma_1^{-1}(\beta_{11}(x) + \beta_{12}(y)), && (x, y) \in X_{11} \times X_{12}\\
F_2(x, y) &= \gamma_2^{-1}(\beta_{21}(x) + \beta_{22}(y)), && (x, y) \in X_{21} \times X_{22}\\
G(x, y) &= \varphi^{-1}(\gamma_1(x) + \gamma_2(y)), && (x, y) \in F_1(X_{11}, X_{12}) \times F_2(X_{21}, X_{22})\\
G_1(x, y) &= \alpha_1^{-1}(\beta_{11}(x) + \beta_{21}(y)), && (x, y) \in X_{11} \times X_{21}\\
G_2(x, y) &= \alpha_2^{-1}(\beta_{12}(x) + \beta_{22}(y)), && (x, y) \in X_{12} \times X_{22}.
\end{aligned}$$

The following theorem also plays an important role in our investigations.

Theorem 1.2 (Maksa [4]). *Let X and Y be intervals of positive length and suppose that $Q : X \times Y \to \mathbb{R}$ is a local quasi-sum on $X \times Y$. Then Q is a quasi-sum on $X \times Y$.*

2. The solution of equation (B) on a restricted domain

Let $X =]a, b[$ and introduce the following notations: $\Delta = \{(x, y) \in X^2 \mid x \le y\}$, $\Delta_c = \{(x, y) \in X^2 \mid x \le y \le c\}$, if $a < c < b$, and $H^* = \{(x, y) \in H \mid x \le y\}$, if $H \subset \mathbb{R}^2$.

Finding the solutions of equation

$$F(G(x,y),G(u,v)) = G(F(x,u),F(y,v)), \tag{BΔ}$$

where (BΔ) holds for all $(x,y) \in \Delta$, $(x,u) \in \Delta$, $(y,v) \in \Delta$, and $(u,v) \in \Delta$, $F : \Delta \to X$ and $G : \Delta \to X$ are CI functions furthermore $F(x,x) = x$ and $G(x,x) = x$ for all $x \in X$ was posed in [6] by M. V. Sokolov in connection with an axiomatization of so-called rank-dependent utility (see Theorem 6 and equation (44) in [6]). The following theorem gives the solutions.

Theorem 2.1. *Let X be an open interval. Suppose that $F : \Delta \to X$ and $G : \Delta \to X$ are CI functions. Then equation* (BΔ) *holds for all $(x,y) \in \Delta$, $(x,u) \in \Delta$, $(y,v) \in \Delta$ and $(u,v) \in \Delta$ if, and only if, there exist CI functions $\varphi : F(X,X) \to \mathbb{R}$ and $\psi : G(X,X) \to \mathbb{R}$ and there exist $\lambda \in \,]0,1[$ and $\mu \in \,]0,1[$ such that*

$$F(x,y) = \varphi^{-1}(\lambda\varphi(x) + (1-\lambda)\varphi(y)), \quad (x,y) \in \Delta, \tag{2.1}$$

$$G(x,y) = \psi^{-1}(\mu\psi(x) + (1-\mu)\psi(y)), \quad (x,y) \in \Delta. \tag{2.2}$$

To prove this theorem we need the following

Lemma 2.2. *Let $a < c < b$, $z_1 \in \,]0,1[$, $z_2 \in \,]0,1[$, let $\delta_1 : \,]a,c] \to \mathbb{R}$ and $\delta_2 : \,]a,c] \to \mathbb{R}$ be CI functions. Equation*

$$\delta_1^{-1}(z_1\delta_1(x) + (1-z_1)\delta_1(y)) = \delta_2^{-1}(z_2\delta_2(x) + (1-z_2)\delta_2(y)) \tag{2.3}$$

holds for all $(x,y) \in \Delta_c$ if, and only if, $z_1 = z_2$ and there exist $0 < \xi \in \mathbb{R}$ and $\eta \in \mathbb{R}$ such that

$$\delta_2(x) = \xi\delta_1(x) + \eta, \quad x \in \,]a,c]. \tag{2.4}$$

Proof. If $\delta_2(x) = \xi\delta_1(x) + \eta$, $x \in \,]a,c]$ for some $0 < \xi \in \mathbb{R}$ and $\eta \in \mathbb{R}$, then an easy calculation gives (2.3).

Now suppose that (2.3) holds for all $(x,y) \in \Delta_c$. Then for the function $\varepsilon = \delta_2 \circ \delta_1^{-1}$ with the notations $p = \delta_1(x)$, and $q = \delta_2(y)$ we have a Jensen equation

$$\varepsilon(z_1p + (1-z_1)q) = z_2\varepsilon(p) + (1-z_2)\varepsilon(q), \quad (p,q) \in \,]\delta_1(a), \delta_1(c)]^{2*}.$$

Applying the method used by Maksa in the proof of the Lemma in [3] we get that there exist $0 < k \in \mathbb{R}$ and $m \in \mathbb{R}$ such that $\varepsilon(r) = \delta_2 \circ \delta_1^{-1}(r) = kr + m$, $r \in \,]\delta_1(a), \delta_1(c)]$ which implies (2.4) and

$$\delta_1^{-1}(z_1\delta_1(x) + (1-z_1)\delta_1(y)) = \delta_1^{-1}(z_2\delta_1(x) + (1-z_2)\delta_1(y)),$$

whence $z_1 = z_2$ follows. $\square$

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. It is easy to check that the functions F and G have the form (2.1) and (2.2) satisfy (BΔ). We have to prove our statement only in the other direction.

Let $a < c < b$ and define the subsets $X_1 = \,]\,a, c]$, $X_2 = [c, b[$ of X and the subsets $X_{11} = \,]\,a, c]^{2*}$, $X_{22} = [c, b[^{2*}$, and $X_{12} = \,]\,a, c] \times [c, b[$ of Δ.

First we show that F and G are quasi-sums on X_{12}. Let $a < d < e < b$. Then (BΔ) holds for all $(x, y) \in \,]\,a, d] \times [d, e]$ and $(u, v) \in [d, e] \times [e, b[$. Thus, by Theorem 1.1, F is a quasi-sum on $]a, d] \times [d, e]$ and G is a quasi-sum on $[d, e] \times [e, b[$ for arbitrary $a < d < e < b$. It is easy to see that functions F and G are local quasi-sums on X_{12}, furthermore, by Theorem 1.2, F and G are quasi-sums on X_{12}.

The generator functions of a CI function are monotonic in the same sense. So, without loss of generality, we may suppose that the generator functions of F and G are CI functions, that is, there exist CI functions $\alpha_1 : X_1 \to \mathbb{R}$, $\beta_1 : X_2 \to \mathbb{R}$, $\gamma_1^{-1} : \alpha_1(X_1) + \beta_1(X_2) \to \mathbb{R}$, $\alpha_2 : X_1 \to \mathbb{R}$, $\beta_2 : X_2 \to \mathbb{R}$, $\gamma_2^{-1} : \alpha_2(X_1) + \beta_2(X_2) \to \mathbb{R}$, such that

$$F(x, y) = \gamma_1^{-1}(\alpha_1(x) + \beta_1(y)), \quad (x, y) \in X_{12}, \tag{2.5}$$

$$G(x, y) = \gamma_2^{-1}(\alpha_2(x) + \beta_2(y), \quad (x, y) \in X_{12}. \tag{2.6}$$

Equation (BΔ) holds for all $(x, u) \in X_{11}$ and $(y, v) \in X_{22}$. It follows from the properties of F that $F(X_{11}) \subset X_1$ and $F(X_{22}) \subset X_2$, so $(x, u) \in X_{11}$, $(y, v) \in X_{22}$ imply that $(x, y) \in X_{12}$, $(u, v) \in X_{12}$, and $(F(x, y), F(u, v)) \in X_{12}$. Thus, by (2.6), (B$\Delta$) can be written in the form

$$\gamma_2 \circ F\big(\gamma_2^{-1}\big(\alpha_2(x) + \beta_2(y)\big), \gamma_2^{-1}\big(\alpha_2(u) + \beta_2(v)\big)\big) = \alpha_2 \circ F(x, u) + \beta_2 \circ F(y, v), \tag{2.7}$$

$(x, u) \in X_{11}, (y, v) \in X_{22}$. With the functions H, K, and L defined by

$$H(t_1, t_2) = \alpha_2 \circ F(\alpha_2^{-1}(t_1), \alpha_2^{-1}(t_2)), \quad (t_1, t_2) \in \alpha_2(X_1)^{2*},$$
$$K(s_1, s_2) = \beta_2 \circ F(\beta_2^{-1}(s_1), \beta_2^{-1}(s_2)), \quad (s_1, s_2) \in \beta_2(X_2)^{2*},$$
$$L(r_1, r_2) = \gamma_2 \circ F(\gamma_2^{-1}(r_1), \gamma_2^{-1}(r_2)), \quad (r_1, r_2) \in (\alpha_2(X_1) + \beta_2(X_2))^{2*}$$

(2.7) goes over into the form

$$L(t_1 + s_1, t_2 + s_2) = H(t_1, t_2) + K(s_1, s_2),$$

$(t_1, t_2) \in \alpha_2(X_1)^{2*}$, $(s_1, s_2) \in \beta_2(X_2)^{2*}$.

Thus, by Theorem 1 in Radó–Baker [5], we have that

$$H(t_1,t_2) = k_1t_1 + k_2t_2 + m_1, \qquad (t_1,t_2) \in \alpha_2(X_1)^{2*},$$
$$K(s_1,s_2) = k_1s_1 + k_2s_2 + m_2, \qquad (s_1,s_2) \in \beta_2(X_2)^{2*},$$
$$L(r_1,r_2) = k_1r_1 + k_2r_1 + m_1 + m_2, \quad (r_1,r_1) \in (\alpha_2(X_1)+\beta_2(X_2))^{2*},$$

where $0 < k_1 \in \mathbb{R}$, $0 < k_2 \in \mathbb{R}$, $m_1 \in \mathbb{R}, m_2 \in \mathbb{R}$. Because of the property $F(x,x) = x$, $x \in X$ we have that $H(x,x) = x$, $x \in \alpha_2(X_1)$ and $K(x,x) = x$, $x \in \beta_2(X_1)$, so, after some calculation, we get that $k_1 + k_2 = 1$ and $m_1 = m_2 = 0$. That is, there exists $w_2 \in\,]0,1[$ such that

$$H(t_1,t_2) = w_2t_1 + (1-w_2)t_2, \quad (t_1,t_2) \in \alpha_2(X_1)^{2*} \tag{2.8}$$

and

$$K(s_1,s_2) = w_2s_1 + (1-w_2)s_2, \quad (s_1,s_2) \in \beta_2(X_2)^{2*}. \tag{2.9}$$

Equations (2.8) and (2.9), with the definition of H and K, imply that

$$F(x,y) = \alpha_2^{-1} \circ G(\alpha_2(x),\alpha_2(y)) = \alpha_2^{-1}(w_2\alpha_2(x) + (1-w_2)\alpha_2(y)),$$
$$(x,y) \in X_{11} \tag{2.10}$$

and

$$F(x,y) = \beta_2^{-1} \circ K(\beta_2(x),\beta_2(y)) = \beta_2^{-1}(w_2\beta_2(x) + (1-w_2)\beta_2(y)), \quad (x,y) \in X_{22},$$

respectively. Let $(c_n) : \mathbb{N} \to X$ be a strictly increasing sequence with limit b. Because of (2.10), we have that for every $n \in \mathbb{N}$ there exist CI functions $\alpha_{c_n} :]\,a,c_n] \to \mathbb{R}$ and $w_{c_n} \in\,]0,1[$ such that

$$F(x,y) = \alpha_{c_n}^{-1}(w_{c_n}\alpha_{c_n}(x) + (1-w_{c_n})\alpha_{c_n}(y)), \quad (x,y) \in \Delta_{c_n}. \tag{2.11}$$

By induction we construct a sequence of CI functions $\varphi_n :]\,a,c_n] \to \mathbb{R}$ $(n \in \mathbb{N})$ with the properties

$$\varphi_n \subset \varphi_{n+1} \quad (n \in \mathbb{N}) \tag{2.12}$$

that is, φ_n is a restriction of φ_{n+1} $(n \in \mathbb{N})$ and

$$F(x,y) = \varphi_n^{-1}(\lambda\varphi_n(x) + (1-\lambda)\varphi_n(y)), \quad (x,y) \in \Delta_{c_n}, \quad (n \in \mathbb{N}), \tag{2.13}$$

where $\lambda \in\,]0,1[$. Let $\varphi_1 = \alpha_{c_1}$ and $\lambda = w_{c_1}$. Then

$$\varphi_1^{-1}(\lambda\varphi_1(x) + (1-\lambda)\varphi_1(y)) = \alpha_{c_2}^{-1}(w_{c_2}\alpha_{c_2}(x) + (1-w_{c_2})\alpha_{c_2}(y)), \quad (x,y) \in \Delta_{c_1}.$$

By our Lemma, there exist $0 < \xi_{c_1} \in \mathbb{R}$ and $\eta_{c_1} \in \mathbb{R}$ such that $\varphi_1(x) = \xi_{c_1}\alpha_{c_2}(x) + \eta_{c_1}$, $x \in]\, a, c_1]$. Let

$$\varphi_2(x) = \xi_{c_1}\alpha_{c_2}(x) + \eta_{c_1}, \quad x \in]\, a, c_2].$$

Then $\varphi_1 \subset \varphi_2$ and $F(x,y) = \varphi_2^{-1}(\lambda\varphi_2(x) + (1-\lambda)\varphi_2(y))$, $(x,y) \in \Delta_{c_2}$.

Continue this procedure. Because of the connection between the functions $\varphi_n :]\, a, c_n] \to \mathbb{R}$ and $\alpha_{c_{n+1}} :]\, a, c_{n+1}] \to \mathbb{R}$ given by our Lemma, we can construct the function φ_{n+1} on $]a, c_{n+1}]$ $(n \in \mathbb{N})$ such that the sequence (φ_n) satisfies (2.12) and (2.13).

Finally define the function $\varphi : X \to \mathbb{R}$ by $\varphi = \bigcup_{n=1}^{\infty} \varphi_n$, that is, for arbitrary $x \in X$ $\varphi(x) = \varphi_n(x)$ for some $n \in \mathbb{N}$. By (2.12), this definition is correct and an easy calculation shows that (2.1) holds.

Because of the symmetry of (BΔ) in F and G, a similar calculation shows that the function G has the form (2.2). □

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